

Photon Hall Effect in Atomic Hydrogen

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(Dated: August 17, 2011)

We predict a photon Hall effect in the optical cross-section of atomic hydrogen, determined by the interference between an electric quadrupole transition and an electric dipole transition from the ground state to $3D_{3/2}$ and $3P_{3/2}$, causing a magneto-transverse acceleration comparable to g . The electric dipole transition only generates a Hall effect when the atom is moving.

Light scattering exchanges momentum between matter and radiation. The momentum of light in matter has been subject to a number of controversies. One recent debate concerns the exchange of momentum with the quantum vacuum [1].

Light scattering is affected by a magnetic field. One feature, the photon Hall effect (PHE), was first predicted in multiple light scattering [2], and observed shortly afterwards [3] with typical changes in the photon flux of $\Delta T_{PHE}/T = 10^{-5}$ per Tesla of applied magnetic field. A Mie theory for the PHE [5] agreed quantitatively with the experiments. Given the wave number \mathbf{k} of the incident photon flux and the magnetic field \mathbf{B} , the PHE induces an exchange of momentum between scatterer and radiation in the magneto-transverse ("upward") direction along $\mathbf{B} \times \mathbf{k}$. A light flux of 10^4 W/m^2 incident on a $10 \mu\text{m}$ particle with a relative PHE of 10^{-5} per Tesla experiences a transverse force of 10^{-19} N/T , roughly equivalent to the Lorentz force on a charge e moving with a velocity of 1 m/s .

Atoms are strong light scatterers with elastic optical cross-sections of order of λ^2 near optical transitions and with promising applications in mesoscopic physics [4]. When the typical Zeeman splitting $\frac{1}{2}\omega_c$ ($\omega_c = eB/m_e = 17.5 \text{ MHz/Gauss}$ is the cyclotron circular frequency) equals the atomic line width ($\gamma \approx 100 \text{ MHz}$), the optical cross-section is significantly altered by the magnetic field, typically true for a few Gauss. Nonetheless, no PHE can occur for pure electric-dipole (ED) transitions, since the ED maintains the symmetry between either forward and backward, or between up and down [5]. To get a nonzero PHE in cold atom gazes, one can consider the scattering from pairs of atoms. For a density of $n = 10^{18}/\text{m}^3$ ^{88}Sr atoms the relative PHE can be as large as a few percent [6].

Can the PHE of a single atom exist at all, and how large will the magneto-transverse momentum transfer to the atom be? We will focus on atomic hydrogen, whose physics in a magnetic field has been studied in great detail [7], with important applications to the formation of the stable spin-polarized phase [8]. The optical

cross-section of an atom is expressed by the Kramers-Heisenberg formula [9],

$$\frac{d\sigma}{d\Omega}(\omega\mathbf{k}\varepsilon \rightarrow \omega_s\mathbf{k}_s\varepsilon_s) = \alpha^2 \frac{\omega_s^3}{\omega^3} |f_{ED}(\omega) + f_{EQ}(\omega) + \dots|^2 \times 1 + \frac{W(\omega_s, \mathbf{k}_s, \varepsilon_s)}{W_0(\omega_s)} \quad (1)$$

Here, α is the fine structure constant, ω and $\omega_s < \omega$ are incident and scattered frequency, ε and ε_s are the polarization vectors of incident and scattered radiation. The last factor accounts for stimulated enhancement (SE) of the cross-section caused by a radiation density W per steradian, per bandwidth, per polarization, with $W_0 = \hbar\omega_s^3/(2\pi c_0)^3$ its value for the quantum vacuum. In this work we will ignore SE and leave the study of "stimulated Hall scattering", likely to become important for $W \approx W_0$, to future work. Finally $f(\omega)$ is the complex scattering amplitude associated with transitions in the atom, that can be either elastic or inelastic. We will concentrate first on the ED transition between the unpolarized singlet ground state $1S_{1/2}$ and the eight $3P_{3/2}$ hyperfine states in atomic hydrogen, for which

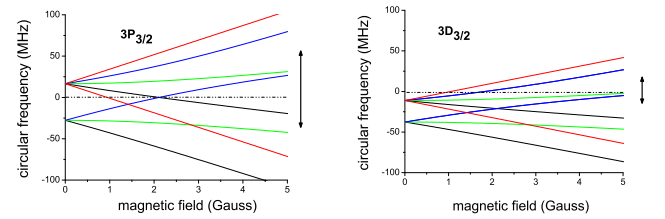


FIG. 1: Hyperfine structure of the $3P_{3/2}$ (left) and $3D_{3/2}$ (right) level of atomic hydrogen, as a function of magnetic field. Equal colors indicate equal values for the hyperfine magnetic quantum number m . The height of the vertical bar on the right indicates the line width γ . The zero in frequency is chosen at the fine structure level of $3P_{3/2}$. The one of $3D_{3/2}$ is 21.07 MHz lower.

$$f_{ED}(\omega) = \frac{\omega^2}{c_0} \sum_{H=1}^8 \frac{(\mathbf{r}_{fH} \cdot \boldsymbol{\varepsilon}_s)(\mathbf{r}_{H0} \cdot \boldsymbol{\varepsilon})}{\omega - \omega_H - i\gamma_P} = \frac{\omega^2 r_{3P}^2}{c_0} \times [P_0(\boldsymbol{\varepsilon}_s \cdot \boldsymbol{\varepsilon}) + [P_1 - P_0](\boldsymbol{\varepsilon}_s \cdot \hat{\mathbf{z}})(\boldsymbol{\varepsilon} \cdot \hat{\mathbf{z}}) + P_2(\omega) i \boldsymbol{\varepsilon}_s \cdot (\boldsymbol{\varepsilon} \times \hat{\mathbf{z}})] \quad (2)$$

Here $\mathbf{r}_{fH} = \langle f | \mathbf{r} | H \rangle$ is the ED matrix element between intermediate and final state, and $r_{3P} = 0.517a_0$ the radial matrix element. In a magnetic field of a few Gauss all hyperfine levels are non-degenerate, with an (anomalous) Zeeman splitting comparable to the hyperfine splitting (Fig 1a). The eigenfunctions $|3P_{j=\frac{3}{2}}\rangle \otimes |f = j \pm \frac{1}{2}, m = -f, \dots, f\rangle$ can be constructed

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\omega \mathbf{k} \rightarrow \omega_s \mathbf{k}_s) &= \frac{\alpha^2 \omega^4 r_{3P}^2}{2c_0^2} \times \\ &|P_0|^2(1 - (\hat{\mathbf{k}}_s \cdot \hat{\mathbf{y}})^2) + |P_1|^2(1 - (\hat{\mathbf{k}}_s \cdot \hat{\mathbf{z}})^2) + |P_2|^2(1 - (\hat{\mathbf{k}}_s \cdot \hat{\mathbf{x}})^2) + 2\text{Im}(P_0 P_2^*)(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{x}})(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{y}}) \end{aligned} \quad (3)$$

The last term in Eq. (3) affects the current along the Hall direction though without inducing a net PHE. This will change when the atom moves. Let K be the frame that moves with the atom, and in which Eq. (3) applies, and K' the one in which the atom moves with velocity \mathbf{v} . The cross-section $d\sigma$ relates an incident flux $\rho(\omega, \mathbf{k})/c_0$ to an outgoing current $\rho_s(\omega_s, \mathbf{k}_s)d\Omega r^2/c_0$ at a distance r in the far field. The latter is unaffected by a Lorentz transformation in order v/c_0 , and since the radiation density ρ transforms as $\rho' = (\omega'^3/\omega^3)\rho$ [10] it follows that,

$$\frac{d\sigma'}{d\Omega'}(\omega' \mathbf{k}' \rightarrow \omega'_s \mathbf{k}'_s) \approx \left(\frac{\omega \omega'_s}{\omega' \omega_s} \right)^3 \frac{d\sigma}{d\Omega}(\omega \mathbf{k} \rightarrow \omega_s \mathbf{k}_s)$$

We insert $\hat{\mathbf{k}}_{(s)} \approx \hat{\mathbf{k}}'_{(s)}(1 + \hat{\mathbf{k}}'_{(s)} \cdot \mathbf{v}/c_0) - \mathbf{v}/c_0$ and $\omega_{(s)} \approx \omega'_{(s)}(1 - \mathbf{v} \cdot \hat{\mathbf{k}}'_{(s)}/c_0)$ and assume for simplicity that the atom moves either parallel or opposite to the incident wave vector \mathbf{k} . The Lorentz transform of the last term in Eq. (3) generates a term $(\mathbf{v} \cdot \hat{\mathbf{x}}')/c_0 [1 - 2(\hat{\mathbf{k}}'_s \cdot \hat{\mathbf{x}}')^2](\hat{\mathbf{k}}'_s \cdot \hat{\mathbf{y}}')$, with $\hat{\mathbf{y}}' = \hat{\mathbf{B}} \times \hat{\mathbf{k}}'$, $\hat{\mathbf{x}}' = \hat{\mathbf{k}}'$ (see Fig. 2). It exhibits a PHE with a momentum transfer to the atom,

$$\begin{aligned} \mathbf{F} &= - \int d^2\Omega' \hbar \mathbf{k}'_s \frac{d\sigma'}{d\Omega'}(\omega' \mathbf{k}' \rightarrow \omega'_s \mathbf{k}'_s) \frac{1}{\hbar \omega'} I(\mathbf{k}') \\ &= \frac{13\pi}{10} \frac{\alpha^2 \omega^4 r_{3P}^4}{c_0^3} \frac{\text{Im}(P_0 P_2^*)}{B} I(\mathbf{k}) \frac{\mathbf{v}}{c_0} \times \mathbf{B} \end{aligned} \quad (4)$$

with $I = 2W c_0 \Delta\omega \Delta\Omega$ the incident flux (W/m^2). This PHE-induced momentum transfer is reminiscent of the magnetic force on a charged particle. We can identify an

from the product states of orbital momentum, electron spin and proton spin with appropriate Clebsch-Gordan coefficients, that constitute the amplitudes P_i , together with the 8 line profiles $f_H(\omega) = 1/(\omega - \omega_H - i\gamma_P)$. We choose $\hat{\mathbf{k}} = \hat{\mathbf{x}}$, $\hat{\mathbf{B}} = \hat{\mathbf{z}}$ and let $\hat{\mathbf{B}} \times \hat{\mathbf{k}} = \hat{\mathbf{y}}$ be the Hall direction. The differential cross-section, averaged over incident polarization, becomes,

induced "virtual charge" as,

$$\begin{aligned} q_v &= \frac{13\pi}{10} \frac{\alpha^2 \omega^4 r_{3P}^4}{c_0^4} \frac{\text{Im}(P_0 P_2^*)}{B} I(\mathbf{k}) \\ &\approx 0.0061 e \frac{\text{Im}(P_0 P_2^*)(\text{MHz})^{-2}}{B(\text{Gauss})} \frac{W}{W_0} \Delta\omega(\text{MHz}) \Delta\Omega(\text{rad}) \end{aligned} \quad (5)$$

This is valid for incident wave vectors either aligned with or opposed to the velocity. One could thus use opposite beams to reduce longitudinal kicks. For this virtual charge to be a genuine charge, an external *electric* field should induce a force $q_v \mathbf{E}$ on the atom at rest. P and T symmetry impose the linear electro cross-section to be of

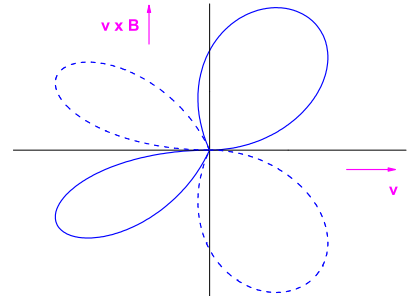


FIG. 2: Polar diagram of the Hall cross-section - last term in Eq. (3) - associated with the ED $3P_{3/2}$ transition of a moving atom with $v = 0.4c_0$ (incident $\mathbf{k} \parallel \mathbf{v}$). Solid indicates increase, dashed a decrease. A net photon current results along $\mathbf{v} \times \mathbf{B}$.

the form $F(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}_s) \mathbf{E} \cdot (\hat{\mathbf{k}} - \hat{\mathbf{k}}_s)$. However, the ED approximation does not allow any odd sequences of \mathbf{k} -vectors. The "induced charge" is thus not a real electromagnetic charge.

Fig. 3a shows that the "induced charge" is of order $10^{-8}e$ per (rad)MHz line width and per steradian (assuming $W = W_0$, $B = 1$ G). When integrated over the full line profile, we obtain a charge $q = -5 \cdot 10^{-8}e$ for 0.1

steradian incident divergence. With this charge, a hydrogen atom at a speed of 10 m/s in a field of 1 Gauss would undergo a transverse deflection of order 4 mm/s². The effect is thus small, and of course it would be interesting to investigate if this effect increases when $W \gg W_0$. This magneto-transverse force vanishes if the incident radiation field is isotropic. In particular, the quantum vacuum does not induce a nett charge.

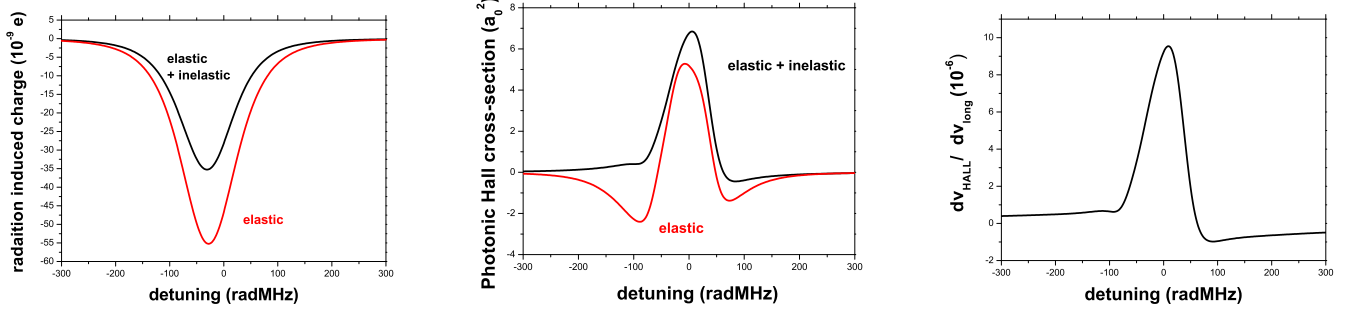


FIG. 3: Left: The virtual charge q_v for the ED transitions from the unpolarized ground state to the $3P_{3/2}$ hyperfine levels in a magnetic field of 1 Gauss (detuning = 0 chosen for the fine structure energy level of $3P_{3/2}$). Red line represents the contribution of the elastic transition; Black line includes the inelastic transitions to the ground state levels with $f = 1$. We adopted $W = W_0$, $\Delta\omega = 1$ MHz and $\Delta\Omega = 1$ sterad. Middle: PHE cross-section from the interference between the ED transition from the spin-singlet ground state $1S$ to the $3P_{3/2}$ level and the EQ transition to the $3D_{3/2}$ level for $B = 5$ G. The red line only counts the elastic transition, the black line includes Raman transitions. Right: Magneto-transverse deflection of the hydrogen atom subject to $B = 5$ G.

The second mechanism that can create a PHE by a single atom is the interference with electric quadrupole (EQ) radiation. Though common in Mie scattering, the strong (hyperfine) splitting and the strict selection rules to excite different multipoles, make such events extremely rare in atomic systems. The line profiles of the hydrogen levels $3P_{3/2}$ and $3D_{3/2}$ of hydrogen are unique in that

they overlap and allow the interference between the ED transition from the unpolarized ground state $1S_{1/2}$ to $3P_{3/2}$ and the EQ transition to $3D_{3/2}$. This gives rise to an interesting "which-way" event inside the hydrogen atom. We adopt the Hall geometry for which $\mathbf{k} \sim \mathbf{x}$ and $\mathbf{B} \sim \mathbf{z}$. For the elastic EQ transition we find,

$$f_{EQ}(\omega) = \frac{1}{4} \frac{\omega^2}{c_0} \sum_{H=1}^8 \frac{(\mathbf{k}_s \cdot \mathbf{r}_{fH})(\mathbf{r}_{fH} \cdot \boldsymbol{\varepsilon}_s)(\mathbf{k} \cdot \mathbf{r}_{H0})(\mathbf{r}_{H0} \cdot \boldsymbol{\varepsilon})}{\omega - \omega_H - i\gamma_D} = \frac{\omega^4 q_{3D}^2}{75 c_0^3} \times$$

$$\sum_{m=\pm 1; i=1,2} Q_1^{(m,i)} \left[(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{r}}_{\pm})(\boldsymbol{\varepsilon}_s \cdot \hat{\mathbf{z}}) + (\hat{\mathbf{k}}_s \cdot \hat{\mathbf{z}})(\boldsymbol{\varepsilon}_s \cdot \hat{\mathbf{r}}_{\pm}) \right] (\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}_{\mp})(\boldsymbol{\varepsilon} \cdot \hat{\mathbf{z}}) + \sum_{m=\pm 2} Q_2^{(m)} (\hat{\mathbf{k}}_s \cdot \hat{\mathbf{r}}_{\pm})(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}_{\mp})(\boldsymbol{\varepsilon}_s \cdot \hat{\mathbf{r}}_{\pm})(\boldsymbol{\varepsilon} \cdot \hat{\mathbf{r}}_{\mp})$$

with $\gamma_D = 32$ MHz the natural line width of the 3D level and $q_{3D} = 0.867 a_0^2$. The summations are over the hyperfine magnetic quantum numbers; the magnetic hyperfine

states $m = \pm 2$ concern only the hyperfine level $f = 2$, so that $Q_2^{(m=\pm 2)} = 1/(\omega - \omega_m - i\gamma_Q)$, whereas the states with $m = \pm 1$ split into two orthogonal superpositions of

$f = 1, 2$ hyperfine levels indicated by the index i . Excited states with $m = 0$ in the EQ amplitude do not contribute to the PHE. The interference with the ED amplitude (2) gives rise to many contributions to the differential cross-

section (summed over the 2 outgoing polarizations, and averaged over the 2 incident polarizations), but not all cross-terms generate a PHE. Those who contribute to the elastic Hall cross-section add up to,

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\text{PHE} - \text{elastic}) = & \frac{\alpha^2 \omega^6 r_{3P}^2 q_{3D}^2}{75 c_0^4} (\hat{\mathbf{k}}_s \cdot \hat{\mathbf{y}}) \times 2\text{Im} \left\{ P_1^* \sum_{i=1,2} \left(Q_1^{(-1,i)} - Q_1^{(1,i)} \right) \frac{1}{2} \left[1 - (\hat{\mathbf{k}}_s \cdot \hat{\mathbf{z}})^2 \right] \right. \\ & \left. + P_0^* \left(Q_2^{(-2)} - Q_2^{(2)} \right) \frac{1}{2} \left[(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{z}})^2 + 2(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{x}})^2 \right] - P_2^* \left(Q_2^{(-2)} + Q_2^{(2)} \right) \frac{1}{2} \left[1 - 2(\hat{\mathbf{k}}_s \cdot \hat{\mathbf{x}})^2 \right] \right\} \quad (6) \end{aligned}$$

The total photonic Hall cross-section σ_H determines the total Hall current in the y -direction. In Figure 3a we show the results for a magnetic field of 5 Gauss. The cross-section is of order a_0^2 as expected from multipolar scaling arguments [9]. The PHE normalized to the ED transverse cross-section is maximally of order $0.1\alpha^2$, and thus only somewhat smaller than the typical values found for Mie scattering at 1 Tesla [5]. Once excited to either $3P_{3/2}$ or $3D_{3/2}$ it is possible for the hydrogen atom to decay inelastically to a polarized hyperfine triplet state. These transitions have been included to obtain Fig. 3b, and slightly change the total Hall cross-section.

Scattering of light transfers momentum to the atom. The dominating ED radiation will accelerate the atom only along the incident radiation. For an incident energy flux $I(\mathbf{k})$ the acceleration is $dv_{\parallel}/dt = I(\mathbf{k})\sigma_{\text{ED}}/mc_0$. The total elastic ED cross-section is easily seen to be $(4\pi/3)(\omega^4 r_{3P}^4 \alpha^2 / c_0^2) \sum_i |P_i|^2$, with a similar contribution from the inelastic transitions. The PHE cross-section induces a magneto-transverse acceleration $dv_{\perp}/dt = I(\mathbf{k})\sigma_{\text{Hall}}(\omega)/mc_0$. For a constant flux $I = W_0 c_0 d\omega d\Omega$ over the line profile we estimate from Fig 3b that $dv_{\perp}/dt \approx I_0(\mathbf{k})600a_0^2 d\Omega/m \approx 10 \text{ m/s}^2$ for 0.1 steradian angular divergence. This equals the gravitational acceleration. One uncertain element in this approach is the large time to accomplish the EQ transition, estimated to be of order $1/\gamma_P \alpha^2 = 200 \mu\text{s}$. The acceleration seems instantaneous at time scales of seconds, but a dynamic theory should be developed to understand this better.

The magneto-transverse deflection can be expressed as $dv_{\perp}/dv_{\parallel} = \sigma_H(\omega)/\sigma_{\text{ED}}(\omega)$, which is independent on radiation intensity, and is shown in Figure 3b as a function of detuning. The inelastic transitions are not detailed but have been included. The deflection is mostly positive (along $\mathbf{B} \times \mathbf{k}$), and of order 10^{-5} near zero detuning. As the atom is accelerated by the longitudinal radiation pressure it will see the light in its rest frame redshifted. If we illuminate at zero detuning an atom initially at rest the transverse speed will reach the value $0.16 \mu\text{m/sec}$, at the moment that $v_{\parallel} = 8 \text{ m/s}$, at a Doppler shift of

roughly five line widths. If we arrange radiation flux and time slot such that the atom stops accelerating at this moment it will have undergone a transverse shift of $0.3 \mu\text{m}$ some 10 meters further on. With a radiation flux of $I = 6 \text{ W/m}^2$ inside a line width of 1 radMHz we avoid stimulated emission and gives us typical time of $t = v_{\parallel} mc_0 / \sigma_{\text{ED}}(0) I = 300 \mu\text{s}$ to accomplish this. This time is larger than the estimated time of $200 \mu\text{s}$ to accomplish the EQ transition.

In conclusion, we have quantified the magneto-transverse scattering of light from unpolarized atomic hydrogen. When moving, the electric dipole transition gives a contribution of order v/c_0 , when at rest the Hall angle is of order α^2 per Gauss, with α the fine structure constant. The study of this effect in the spin-polarized S -state of atomic hydrogen [8] is a next step. This work was supported by the ANR contract PHOTONIMPULS ANR-09-BLAN-0088-01.

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